Matrix-Vector Multiplication

(Section 2.2)

Recall: how to show one vector is a linear combination of other, given vectors – we expanded all the operations, rewrote as a system of linear equations, then used a matrix to solve the system. We would like to go straight to the matrix!

Rewrite using Matrix-Vector Multiplication

Let $\vec{v}$ be a vector in $\mathbb{R}^n$ and

let A be an $m \times n$ matrix.

Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m$ denote the

row vectors of A, so

$$A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix}.$$

Each $\vec{a}_i$ for

$1 \leq i \leq m$ is a vector in $\mathbb{R}^n$, so

we can dot-product $\vec{a}_i$ with v.

We then write the <u>matrix-vector product</u> of A and $\vec{v}$ by

$$\boxed{A \cdot \vec{v} = \begin{bmatrix} \vec{a}_1 \cdot \vec{v} \\ \vec{a}_2 \cdot \vec{v} \\ \vec{a}_3 \cdot \vec{v} \\ \vdots \\ \vec{a}_m \cdot \vec{v} \end{bmatrix}}\,,$$

a vector in $\mathbb{R}^m$ (each dot product is a real number, we have m of them).

Example 1: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$. Compute

$$A \cdot \vec{v}.$$

Solution:

$$A \cdot \vec{v} = \begin{bmatrix} [1 \;\; 2] \cdot \begin{bmatrix} -5 \\ 6 \end{bmatrix} \\ [3 \;\; 4] \cdot \begin{bmatrix} -5 \\ 6 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

Observations:

1) Dimension match: If A is an $m \times n$ matrix and $\vec{v}$ is a $k \times 1$ vector, then the product

$$\underset{(m \times n)}{A} \cdot \underset{(k \times 1)}{\vec{v}}$$ makes sense only when $n = k$.

equal

The output $A\vec{v}$ is an $m \times 1$ vector.

2) Matrix columns: If A is an $m \times n$ matrix, let $\vec{e}_1$ in $\mathbb{R}^n$ be the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

i.e. $\vec{e}_1$ is the vector with a 1 in the first entry, zeros in all other entries. What happens with the product $A \cdot \vec{e}_1$?

Example: $A = \begin{bmatrix} 2 & 1 & -5 \\ 6 & 0 & 3 \end{bmatrix}$

A is 2×3

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ 3×1

$$\underset{(2\times3)(3\times1)}{A \cdot \vec{e}_1} = \begin{bmatrix} [2 \ 1 \ -5] \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ [6 \ 0 \ 3] \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$A \cdot \vec{e}_1 = \begin{bmatrix} [2 \ \ 1 \ \ -5] \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ [6 \ \ 0 \ \ 3] \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$ the first column of A

$$A = \begin{bmatrix} 2 & 1 & -5 \\ 6 & 0 & 3 \end{bmatrix}$$

In general, $A \cdot \vec{e}_1 =$ the first column of A

If $\vec{e}_2$ is the vector in $\mathbb{R}^n$ with a one in the second entry, zeros in all other entries, then

$$A \cdot \vec{e}_2 = \text{the second column of } A$$

For our example,

$$\begin{bmatrix} 2 & 1 & -5 \\ 6 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} [2 \ 1 \ -5] \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ [6 \ 0 \ 3] \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \checkmark$$

If $\vec{e}_3$ is the vector in $\mathbb{R}^n$ with a 1 in the 3^{rd} entry and zeros in all other entries,

$$A \cdot \vec{e}_3 = \text{the third column of } A$$

In general, if e_k is the vector in $\mathbb{R}^n$ with a 1 in the k^{th} entry and zeros in all other entries,

$$A \cdot \vec{e}_k = \text{the } k^{th} \text{ column of } A$$

(there are n $\vec{e}_k$ vectors in $\mathbb{R}^n$)

Example 2: Let T be the function that takes a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ to the vector $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$. Express T as a matrix.

Solution: Suppose A is the matrix that implements the function T; that is,

$$\underset{(m \times n)}{A} \underset{3 \times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{3 \times 1}{\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}}$$

Find an explicit description of A.

$A\vec{e}_1$ = the first column of A.

But since $A\vec{e}_1 = A\begin{bmatrix}1\\0\\0\end{bmatrix}$ is

supposed to equal $\begin{bmatrix}x\\y\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$,

the first column of A is $\begin{bmatrix}1\\0\\0\end{bmatrix}$.

$$\underbrace{A}_{= 3\times 3}(3\times 1) = 3\times 1$$

Since A is 3×3, we need two more columns.

$A\vec{e}_2$ = second column of A.

$x = z = 0$
$y = 1$

$A\vec{e}_2 = A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is supposed to

equal $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, so

the second column of A is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Finally, $A\vec{e}_3$ = third column of A,

$x = y = 0$, $z = 1$

$A\vec{e}_3 = A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is supposed to

equal $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Is this correct?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} [1\ 0\ 0] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ [0\ 1\ 0] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ [0\ 0\ 0] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

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